Integral functions

Frames 1 to 79

Learning outcomes

When you have completed this Programme you will be able to:

- Derive the recurrence relation for the gamma function and evaluate the gamma function for certain rational arguments
- Evaluate integrals that require the use of the gamma function in their solution
- Identify the beta function and evaluate integrals that require the use of the beta function in their solution
- Derive the relationship between the gamma function and the beta function
- Use the duplication formula to evaluate the gamma function for half integer arguments
- Recognise the error function and its relation to the Gaussian probability distribution
- Recognise elliptic functions of the first and second kind
- Evaluate integrals that require the use of elliptic functions in their solution
- Use alternative forms of the elliptic functions

Prerequisite: Engineering Mathematics (Fifth Edition)
Programmes 15 Integration 1, 16 Integration 2 and 17 Reduction formulas

Integral functions

1

Some functions are most conveniently defined in the form of integrals and we shall deal with one or two of these in the present Programme.

The gamma function

The gamma function $\Gamma(x)$ is defined by the integral

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\,dt \qquad (1)$$

and is convergent for $x > 0$.

From (1): $\Gamma(x+1) = \int_0^{\infty} t^x e^{-t} dt$

Integrating by parts

$$\Gamma(x+1) = \left[t^x\left(\frac{e^{-t}}{-1}\right)\right]_0^{\infty} + x\int_0^{\infty} e^{-t}t^{x-1}\,dt$$

$$= \{0-0\} + x\Gamma(x)$$

$$\therefore\ \Gamma(x+1) = x\Gamma(x) \qquad (2)$$

This is a fundamental recurrence relation for gamma functions. It can also be written as $\Gamma(x) = (x-1)\Gamma(x-1)$

With it we can derive a number of other results.

For instance, when $x = n$, a positive integer ≥ 1, then

$\Gamma(n+1) = n\Gamma(n)$ But $\Gamma(n) = (n-1)\Gamma(n-1)$

$= n(n-1)\Gamma(n-1)$ $\Gamma(n-1) = (n-2)\Gamma(n-2)$

$= n(n-1)(n-2)\Gamma(n-2)$

— — — — — —

$= n(n-1)(n-2)(n-3)\ldots 1\Gamma(1) = n!\Gamma(1)$

But, from the original definition $\Gamma(1) = \ldots\ldots\ldots\ldots$

2

$$\boxed{\Gamma(1) = 1}$$

Because

$$\Gamma(1) = \int_0^{\infty} t^0 e^{-t} dt = \Big[-e^{-t}\Big]_0^{\infty} = 0 + 1 = 1$$

Therefore, we have $\Gamma(1) = 1$ (3)

and $\Gamma(n+1) = n!$ provided n is a positive integer.

$\therefore\ \Gamma(7) = \ldots\ldots\ldots\ldots$

3

$$\boxed{\Gamma(7) = 720}$$

Because

$\Gamma(7) = \Gamma(6+1) = 6! = 720.$

Knowing $\Gamma(7) = 720$, $\Gamma(8) = \ldots\ldots\ldots\ldots$ and $\Gamma(9) = \ldots\ldots\ldots\ldots$

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$$\boxed{\Gamma(8) = 5040; \quad \Gamma(9) = 40\,320}$$

Because

$$\Gamma(8) = \Gamma(7+1) = 7\Gamma(7) = 7(720) \quad = 5040$$
$$\Gamma(9) = \Gamma(8+1) = 8\Gamma(8) = 8(5040) = 40\,320$$

We can also use the recurrence relation in reverse

$$\Gamma(x+1) = x\Gamma(x)$$
$$\therefore \; \Gamma(x) = \frac{\Gamma(x+1)}{x} \qquad (4)$$

For example, given that $\Gamma(7) = 720$, we can determine $\Gamma(6)$

$$\Gamma(6) = \frac{\Gamma(6+1)}{6} = \frac{\Gamma(7)}{6} = \frac{720}{6} = 120$$

and then $\Gamma(5) = \ldots\ldots\ldots\ldots$

5

$$\boxed{\Gamma(5) = 24}$$

$$\Gamma(5) = \frac{\Gamma(5+1)}{5} = \frac{\Gamma(6)}{5} = \frac{120}{5} = 24.$$

So far, we have used the original definition

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$$

for cases where x is a positive integer n.

What happens when $x = \frac{1}{2}$? We will investigate.

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty t^{-1/2}e^{-t}\,dt$$

Putting $t = u^2$, $dt = 2u\,du$, then

$$\Gamma(\tfrac{1}{2}) = \ldots\ldots\ldots\ldots$$

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$$\boxed{\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-u^2}\,du}$$

Because

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty u^{-1}e^{-u^2}2u\,du = 2\int_0^\infty e^{-u^2}\,du.$$

Unfortunately, $\int_0^\infty e^{-u^2}\,du$ cannot easily be determined by normal means. It is, however, important, so we have to find a way of getting round the difficulty.

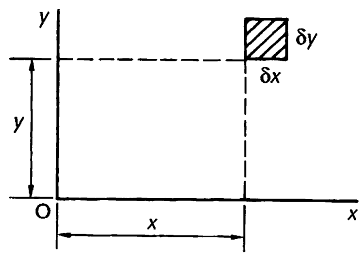
▶

Evaluation of $\int_0^{\infty} e^{-x^2}\,dx$

Let $I = \int_0^{\infty} e^{-x^2}\,dx$, then also $I = \int_0^{\infty} e^{-y^2}\,dy$

$$\therefore\ I^2 = \left(\int_0^{\infty} e^{-x^2}dx\right)\left(\int_0^{\infty} e^{-y^2}dy\right) = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

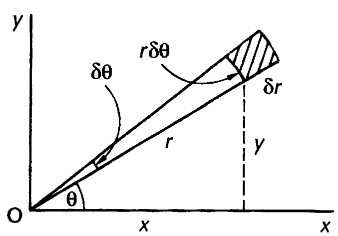
$\delta a = \delta x\,\delta y$ represents an element of area in the x–y plane and the integration with the stated limits covers the whole of the first quadrant.



Converting to polar coordinates, the element of area $\delta a = r\,\delta\theta\,\delta r$. Also, $x^2 + y^2 = r^2$

$$\therefore\ e^{-(x^2+y^2)} = e^{-r^2}$$

For the integration to cover the same region as before,



the limits of r are $r = 0$ to $r = \infty$
the limits of θ are $\theta = 0$ to $\theta = \pi/2$.

$$\therefore\ I^2 = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta = \int_0^{\pi/2}\left[-\frac{e^{-r^2}}{2}\right]_0^{\infty} d\theta$$

$$= \int_0^{\pi/2}\left(\frac{1}{2}\right) d\theta = \left[\frac{\theta}{2}\right]_0^{\pi/2} = \frac{\pi}{4}$$

$$\therefore\ I = \frac{\sqrt{\pi}}{2}$$

$$\therefore\ \int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} \qquad (5)$$

This result opens the way for others, so make a note of it and then move on to the next frame

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Before that diversion, we had established that

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-u^2}\,du$$

We now know that $\displaystyle\int_0^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2} \qquad \therefore\ \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

From this, using the recurrence relation $\Gamma(x+1) = x\Gamma(x)$, we can obtain the following

$$\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\ \Gamma(\tfrac{1}{2}) = \tfrac{1}{2}(\sqrt{\pi}) \qquad \therefore\ \Gamma(\tfrac{3}{2}) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(\tfrac{5}{2}) = \tfrac{3}{2}\ \Gamma(\tfrac{3}{2}) = \tfrac{3}{2}\left(\frac{\sqrt{\pi}}{2}\right) \qquad \therefore\ \Gamma(\tfrac{5}{2}) = \frac{3\sqrt{\pi}}{4}$$

$$\Gamma(\tfrac{7}{2}) = \dots\dots\dots\dots$$

8

$$\boxed{\Gamma(\tfrac{7}{2}) = \frac{15\sqrt{\pi}}{8}}$$

Because

$$\Gamma(\tfrac{7}{2}) = \Gamma(\tfrac{5}{2}+1) = \tfrac{5}{2}\ \Gamma(\tfrac{5}{2}) = \frac{5}{2}\left(\frac{3\sqrt{\pi}}{4}\right) = \frac{15\sqrt{\pi}}{8}$$

Using the recurrence relation in reverse, i.e. $\Gamma(x) = \dfrac{\Gamma(x+1)}{x}$, we can also obtain

$$\Gamma(-\tfrac{3}{2}) = \frac{\Gamma(-\frac{1}{2})}{-\frac{3}{2}} = \frac{\Gamma(\frac{1}{2})}{(-\frac{3}{2})(-\frac{1}{2})} = \tfrac{4}{3}\sqrt{\pi}$$

Negative values of x

Since $\Gamma(x) = \dfrac{\Gamma(x+1)}{x}$, then as $x \to 0$, $\Gamma(x) \to \infty \quad \therefore\ \Gamma(0) = \infty$.

The same result occurs for all negative integral values of x – which does not follow from the original definition, but which is obtainable from the recurrence relation.

Because at $x = -1, \quad \Gamma(-1) = \dfrac{\Gamma(0)}{-1} = \infty$

$x = -2, \quad \Gamma(-2) = \dfrac{\Gamma(-1)}{-2} = \infty$ etc.

Also, at $x = -\tfrac{1}{2}, \quad \Gamma(-\tfrac{1}{2}) = \dfrac{\Gamma(\frac{1}{2})}{-\frac{1}{2}} = -2\sqrt{\pi}$

and at $x = -\tfrac{3}{2}, \quad \Gamma(-\tfrac{3}{2}) = \dfrac{\Gamma(-\frac{1}{2})}{-\frac{3}{2}} = \dfrac{4}{3}\sqrt{\pi}$

Similarly $\Gamma(-\tfrac{5}{2}) = \dots\dots\dots\dots$

and $\Gamma(-\tfrac{7}{2}) = \dots\dots\dots\dots$

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$$\boxed{\Gamma(-\tfrac{5}{2}) = -\frac{8}{15}\sqrt{\pi};\quad \Gamma(-\tfrac{7}{2}) = \frac{16}{105}\sqrt{\pi}}$$

So we have

(a) For n a positive integer

$\Gamma(n+1) = n\Gamma(n) = n!$

$\Gamma(1) = 1;\quad \Gamma(0) = \infty;\quad \Gamma(-n) = \pm\infty$

(b) $\Gamma(\tfrac{1}{2}) = \sqrt{\pi};\qquad \Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi}$

$\Gamma(\tfrac{3}{2}) = \dfrac{\sqrt{\pi}}{2};\qquad \Gamma(-\tfrac{3}{2}) = \dfrac{4}{3}\sqrt{\pi}$

$\Gamma(\tfrac{5}{2}) = \dfrac{3\sqrt{\pi}}{4};\qquad \Gamma(-\tfrac{5}{2}) = -\dfrac{8}{15}\sqrt{\pi}$

$\Gamma(\tfrac{7}{2}) = \dfrac{15\sqrt{\pi}}{8};\qquad \Gamma(-\tfrac{7}{2}) = \dfrac{16}{105}\sqrt{\pi}$

This is quite a useful list. Make a note of it for future use

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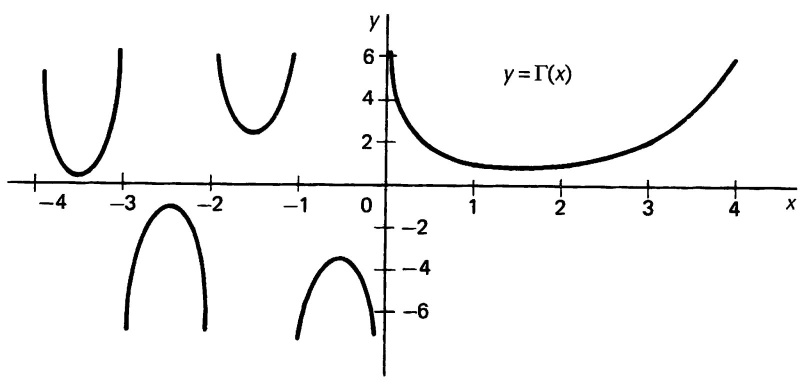
Graph of $y = \Gamma(x)$

Values of $\Gamma(x)$ for a range of positive values of x are available in tabulated form in various sets of mathematical tables. These, together with the results established above, enable us to draw the graph of $y = \Gamma(x)$.

x	0	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0
$\Gamma(x)$	∞	1·772	1·000	0·886	1·000	1·329	2·000	3·323	6·000

x	−0·5	−1·5	−2·5	−3·5
$\Gamma(x)$	−3·545	2·363	−0·945	0·270



For large n it can be shown that $\Gamma(n+1) \approx \sqrt{2\pi n}\, n^n\, e^{-n}$ which gives rise to Stirling's formula for an approximation to the factorial of a large number

$n! \approx \sqrt{2\pi n}\, n^n\, e^{-n}$

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Revision

Let us now revise the main points before we move on to some examples.

The definition of $\Gamma(x)$ is that $\Gamma(x) =$ ....

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$$\boxed{\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t}$$

The recurrence relation states that

$$\Gamma(x+1) = \ldots\ldots\ldots$$

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$$\boxed{\Gamma(x+1) = x\Gamma(x)}$$

When x is a positive integer, i.e. $x = n$, then

$$\Gamma(n+1) = \ldots\ldots\ldots$$

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$$\boxed{\Gamma(n+1) = n!}$$

Then we have a number of specific results

$$\Gamma(1) = \ldots\ldots\ldots; \quad \Gamma(0) = \ldots\ldots\ldots; \quad \Gamma(\tfrac{1}{2}) = \ldots\ldots\ldots$$

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$$\boxed{\Gamma(1) = 1; \quad \Gamma(0) = \infty; \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}}$$

and finally, for all negative integral values of n

$$\Gamma(n) = \ldots\ldots\ldots$$

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$$\boxed{\Gamma(n) = \pm\infty}$$

Listing them together, we have

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t$$

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma(n+1) = n! \qquad \text{for } n \text{ a positive integer}$$

$$\Gamma(1) = 1; \qquad \Gamma(0) = \infty; \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(n) = \pm\infty \quad \text{for } n \text{ a negative integer.}$$

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Now for a few examples of evaluation of integrals.

Example 1

Evaluate $\int_0^\infty x^7 e^{-x} dx$.

We recognise this as the standard form of the gamma function

$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ with the variables changed.

It is often convenient to write the gamma function as

$$\Gamma(v) = \int_0^\infty x^{v-1} e^{-x} dx$$

Our example then becomes

$$I = \int_0^\infty x^7 e^{-x} dx = \int_0^\infty x^{v-1} e^{-x}\, dx \qquad \text{where } v = \ldots\ldots\ldots\ldots$$

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$$\boxed{v = 8}$$

$$\therefore\ I = \Gamma(v) = \Gamma(8) = \ldots\ldots\ldots\ldots$$

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$$\boxed{\Gamma(8) = 7! = 5040}$$

$$\text{i.e. } \int_0^\infty x^7\, e^{-x}\, dx = \Gamma(8) = 7! = 5040$$

Example 2

Evaluate $\int_0^\infty x^3\, e^{-4x}\, dx$.

If we compare this with $\Gamma(v) = \int_0^\infty x^{v-1}\, e^{-x}\, dx$, we must reduce the power of e to a single variable, i.e. put $y = 4x$, and we use this substitution to convert the whole integral into the required form.

$$y = 4x \qquad \therefore\ dy = 4\, dx \qquad \text{Limits remain unchanged.}$$

The integral now becomes ....

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$$\boxed{I = \int_0^\infty \left(\frac{y}{4}\right)^3 e^{-y} \frac{dy}{4}}$$

$$\therefore\ I = \frac{1}{4^4} \int_0^\infty y^3\, e^{-y}\, dy = \frac{1}{4^4}\Gamma(v) \qquad \text{where } v = \ldots\ldots\ldots\ldots$$

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$$\boxed{v = 4}$$

Because

$$\int_0^\infty y^{v-1}e^{-y}\mathrm{d}y = \int_0^\infty y^3 e^{-y}\mathrm{d}y \qquad \therefore\ v = 4$$

$$\therefore\ I = \frac{1}{4^4}\,\Gamma(4) = \ldots\ldots\ldots\ldots$$

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$$\boxed{I = \frac{3}{128}}$$

Because

$$I = \frac{1}{256}\Gamma(4) = \frac{1}{256}(3!) = \frac{6}{256} = \frac{3}{128}$$

One more.

Example 3

Evaluate $\displaystyle\int_0^\infty x^{1/2}\, e^{-x^2}\, \mathrm{d}x$.

The substitution here is to put ....

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$$\boxed{y = x^2}$$

Work through it as before. When you have completed it, check with the next frame.

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Here is the working.

$y = x^2 \quad \therefore\ \mathrm{d}y = 2x\,\mathrm{d}x$ Limits $x = 0, y = 0;\quad x = \infty, y = \infty.$

$x = y^{1/2} \quad \therefore\ x^{1/2} = y^{1/4}$

$$\therefore\ I = \int_0^\infty y^{1/4}\, e^{-y}\, \mathrm{d}y/2x = \int_0^\infty \frac{y^{1/4}\, e^{-y}\, \mathrm{d}y}{2y^{1/2}}$$

$$= \frac{1}{2}\int_0^\infty y^{-1/4}\, e^{-y}\, \mathrm{d}y$$

$$= \frac{1}{2}\int_0^\infty y^{v-1}\, e^{-y}\mathrm{d}y \quad \text{where } v = \frac{3}{4} \qquad \therefore\ I = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)$$

From tables, $\Gamma(0{\cdot}75) = 1{\cdot}2254$

$\therefore\ I = 0{\cdot}613$

▶

Here is part of a table that may be useful.

x	$\Gamma(x)$
0·25	3·6256
0·50	1·7725
0·75	1·2254
1·00	1·0000
1·25	0·9064
1·50	0·8862
1·75	0·9191
2·00	1·0000
2·25	1·1330
2·50	1·3293

x	$\Gamma(x)$
2·75	1·6084
3·00	2·0000
3·25	2·5493
3·50	3·3234
3·75	4·4230
4·00	6·0000
4·25	8·2851
4·50	11·6318
4·75	16·5862
5·00	24·0000

Now we will move on to another set of functions closely related to gamma functions.

Let us start a new frame

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The beta function

The beta function $\mathrm{B}(m, n)$, is defined by

$$\mathrm{B}(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x \qquad (1)$$

which converges for $m > 0$ and $n > 0$.

Putting $(1-x) = u \quad \therefore\ x = 1-u \quad \therefore\ \mathrm{d}x = -\,\mathrm{d}u$

Limits: when $x = 0,\ u = 1;\quad$ when $x = 1,\ u = 0$

$$\therefore\ \mathrm{B}(m, n) = -\int_1^0 (1-u)^{m-1}\,u^{n-1}\,\mathrm{d}u = \int_0^1 (1-u)^{m-1}\,u^{n-1}\mathrm{d}u$$

$$= \int_0^1 u^{n-1}(1-u)^{m-1}\mathrm{d}u = \mathrm{B}(n, m)$$

$$\therefore\ \mathrm{B}(m, n) = \mathrm{B}(n, m) \qquad (2)$$

Alternative form of the beta function

We had

$$\mathrm{B}(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x$$

If we put $x = \sin^2\theta$, the result then becomes ....

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$$\boxed{\mathrm{B}(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,\mathrm{d}\theta}$$

Because if $x = \sin^2\theta$, $\mathrm{d}x = 2\sin\theta\,\cos\theta\,\mathrm{d}\theta$.

When $x = 0$, $\theta = 0$; when $x = 1$, $\theta = \pi/2$. $1 - x = 1 - \sin^2\theta = \cos^2\theta$

$$\therefore\ \mathrm{B}(m, n) = 2\int_0^{\pi/2} \sin^{2m-2}\theta\cos^{2n-2}\theta\sin\theta\,\cos\theta\,\mathrm{d}\theta$$

$$\therefore\ \mathrm{B}(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,\mathrm{d}\theta \qquad (3)$$

Make a note of this result. We shall need to use it later.

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Reduction formulas

In Programme 17 of *Engineering Mathematics (Fifth Edition)* we established useful reduction formulas relating to integrals of powers of sines and cosines, particularly when the integral limits are 0 and $\pi/2$.

(a) $$\int_0^{\pi/2} \sin^n x\,\mathrm{d}x = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,\mathrm{d}x \quad \text{i.e. } S_n = \frac{n-1}{n}S_{n-2} \qquad (4)$$

(b) $$\int_0^{\pi/2} \cos^n x\,\mathrm{d}x = \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\,\mathrm{d}x \quad \text{i.e. } C_n = \frac{n-1}{n}C_{n-2} \qquad (5)$$

A third reduction formula for products of powers of sines and cosines is

(c) $$\int_0^{\pi/2} \sin^m x\,\cos^n x\,\mathrm{d}x = \frac{m-1}{m+n}\int_0^{\pi/2} \sin^{m-2} x\,\cos^n x\,\mathrm{d}x$$

If we denote $\int_0^{\pi/2} \sin^m x\,\cos^n x\,\mathrm{d}x$ by $I_{m,\,n}$, the last result can be written

$$I_{m,\,n} = \frac{m-1}{m+n}I_{m-2,\,n} \qquad (6)$$

Alternatively, $\int_0^{\pi/2} \sin^m x\,\cos^n x\,\mathrm{d}x$ can be expressed as

$$\frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x\,\cos^{n-2} x\,\mathrm{d}x$$

$$\text{i.e.}\quad I_{m,\,n} = \frac{n-1}{m+n}I_{m,\,n-2} \qquad (7)$$

▶

Now $B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$ and if we apply (6) to the integral, we have

$$\begin{aligned}
&\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta \\
&= \frac{(2m-1)-1}{(2m-1)+(2n-1)}\int_0^{\pi/2} \sin^{2m-3}\theta\,\cos^{2n-1}\theta\,d\theta \\
&= \frac{m-1}{m+n-1}\int_0^{\pi/2} \sin^{2m-3}\theta\cos^{2n-1}\theta\,d\theta
\end{aligned}$$

Now, using (7) with the right-hand integral

$$\begin{aligned}
\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta &= \frac{m-1}{m+n-1}\cdot\frac{(2n-1)-1}{(2m-3)+(2n-1)} \\
&\quad\times\int_0^{\pi/2} \sin^{2m-3}\theta\cos^{2n-3}\theta\,d\theta \\
&= \frac{m-1}{m+n-1}\cdot\frac{n-1}{m+n-2} \\
&\quad\times\int_0^{\pi/2} \sin^{2m-3}\theta\cos^{2n-3}\theta\,d\theta
\end{aligned}$$

$$\therefore\quad B(m, n) = \frac{(m-1)(n-1)}{(m+n-1)(m+n-2)}\cdot 2\int_0^{\pi/2} \sin^{2m-3}\theta\,\cos^{2n-3}\theta\,d\theta$$

$$\text{i.e. } B(m, n) = \frac{(m-1)(n-1)}{(m+n-1)(m+n-2)}B(m-1, n-1) \qquad (8)$$

This is obviously a reduction formula for $B(m, n)$ and the process can be repeated as required.

For example $B(4, 3) = \ldots\ldots\ldots\ldots$

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$$\boxed{B(4, 3) = \frac{(3)(2)}{(6)(5)}\frac{(2)(1)}{(4)(3)}B(2, 1)}$$

Because, applying (8)

$$B(4, 3) = \frac{(3)(2)}{(6)(5)}B(3, 2) = \frac{(3)(2)}{(6)(5)}\frac{(2)(1)}{(4)(3)}B(2, 1)$$

Now we must evaluate $B(2, 1)$ for we can go no further in the reduction process, since, from the definition of $B(m, n)$, m and n must be

$\ldots\ldots\ldots\ldots$

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$$\boxed{>0}$$

But $B(2,\ 1) = 2\int_0^{\pi/2} \sin^3\theta\ \cos\theta\ d\theta = 2\left[\dfrac{\sin^4\theta}{4}\right]_0^{\pi/2} = \dfrac{1}{2}$

$$\therefore\ B(4,\ 3) = \frac{(3)(2)}{(6)(5)}\frac{(2)(1)}{(4)(3)}\frac{1}{2}$$

$$= \frac{(3)(2)(1)\times(2)(1)}{(6)(5)(4)(3)(2)(1)} = \frac{(3!)(2!)}{(6!)}$$

Similarly, $B(5,\ 3) = \dots\dots\dots\dots$

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$$\boxed{B(5,3) = \frac{(4!)(2!)}{(7!)}}$$

Because

$$B(5,3) = \frac{(4)(2)}{(7)(6)}B(4,2) = \frac{(4)(2)}{(7)(6)}\frac{(3)(1)}{(5)(4)}B(3,1)$$

$$B(3,1) = 2\int_0^{\pi/2} \sin^5\theta\ \cos\theta\ d\theta = 2\left[\frac{\sin^6\theta}{6}\right]_0^{\pi/2} = \frac{1}{3}$$

$$\therefore\ B(5,3) = \frac{(4)(2)}{(7)(6)}\frac{(3)(1)}{(5)(4)}\frac{1}{3}\frac{(2)}{(2)} = \frac{(4!)(2!)}{(7!)}$$

In general $B(m,\ n) = \dfrac{(m-1)!(n-1)!}{(m+n-1)!}$ (9)

Note that $B(k,\ 1) = 2\int_0^{\pi/2} \sin^{2k-1}\theta\ \cos\theta\ d\theta$

$$= 2\int_0^{\pi/2} \sin^{2k-1}\theta\ d(\sin\theta)$$

$$= 2\left[\frac{\sin^{2k}\theta}{2k}\right]_0^{\pi/2} = \frac{1}{k}$$

$$\therefore\ B(k,\ 1) = \frac{1}{k}$$

$$\therefore\ B(k,\ 1) = B(1,\ k) = \frac{1}{k} \qquad (10)$$

We can also use the trigonometrical definition (3) to evaluate $B(\frac{1}{2},\frac{1}{2})$

$$B(\tfrac{1}{2},\tfrac{1}{2}) = \dots\dots\dots\dots$$

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$$\boxed{\mathrm{B}(\tfrac{1}{2},\tfrac{1}{2}) = \pi}$$

Because

$$\mathrm{B}(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, \mathrm{d}\theta$$

$$\therefore \mathrm{B}(\tfrac{1}{2}, \tfrac{1}{2}) = 2\int_0^{\pi/2} \sin^0\theta \cos^0\theta \, \mathrm{d}\theta$$

$$= 2\int_0^{\pi/2} 1 \, \mathrm{d}\theta = 2\Big[\theta\Big]_0^{\pi/2} = \pi \qquad (11)$$

Now let us summarise our various results so far.

Next frame

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Revision

$$\mathrm{B}(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} \, \mathrm{d}x \qquad m > 0,\ n > 0$$

$$\mathrm{B}(m, n) = \mathrm{B}(n, m)$$

$$\mathrm{B}(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, \mathrm{d}\theta$$

$$\mathrm{B}(m, n) = \frac{(m-1)(n-1)}{(m+n-1)(m+n-2)}\mathrm{B}(m-1,\ n-1)$$

$$\mathrm{B}(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!} \qquad m \text{ and } n \text{ positive integers}$$

$$\mathrm{B}(k, 1) = \mathrm{B}(1, k) = \frac{1}{k} \qquad \therefore \mathrm{B}(1, 1) = 1$$

$$\mathrm{B}(\tfrac{1}{2}, \tfrac{1}{2}) = \pi$$

Be sure that you are familiar with all these. We shall be using them all in due course.

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Relation between the gamma and beta functions

If m and n are positive integers

$$\mathrm{B}(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

Also, we have previously established that, for n a positive integer,

$$n! = \Gamma(n+1)$$

$$\therefore (m-1)! = \Gamma(m) \quad \text{and} \quad (n-1)! = \Gamma(n)$$

and also $(m+n-1)! = \Gamma(m+n)$

$$\therefore \mathrm{B}(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \qquad (12)$$

The relation $\mathrm{B}(m, n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ holds good even when m and n are not necessarily integers.

We will prove this in the next frame, so move on

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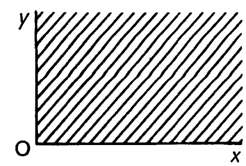
Proof that $\mathrm{B}(m, n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Let $\Gamma(m) = \int_0^\infty x^{m-1}\, e^{-x}\, dx$ and $\Gamma(n) = \int_0^\infty y^{n-1}\, e^{-y}\, dy$

$$\therefore\ \Gamma(m)\Gamma(n) = \int_0^\infty x^{m-1}\, e^{-x}\, dx \int_0^\infty y^{n-1}\, e^{-y}\, dy$$
$$= \int_0^\infty \int_0^\infty x^{m-1}\, y^{n-1}\, e^{-(x+y)}\, dx\, dy$$

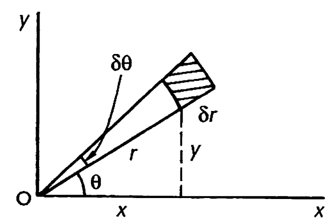


Note that the integration is carried out over the first quadrant of the x–y plane.

Putting $x = u^2$ and $y = v^2$ $\quad dx = 2u\,du$ and $\quad dy = 2v\,dv$

$$\therefore\ \Gamma(m)\Gamma(n) = 4\int_0^\infty \int_0^\infty u^{2m-2}\, v^{2n-2}\, e^{-(u^2+v^2)} uv\, du\, dv$$
$$= 4\int_0^\infty \int_0^\infty u^{2m-1}\, v^{2n-1}\, e^{-(u^2+v^2)}\, du\, dv$$

If we now convert to polar coordinates,
$u = r\cos\theta$; $\quad v = r\sin\theta$; $du\, dv = r\, dr\, d\theta$
$u^2 + v^2 = r^2 \quad 0 < r < \infty \quad 0 < \theta < \pi/2$



$$\therefore\ \Gamma(m)\Gamma(n) = 4\int_0^{\pi/2} \int_0^\infty r^{2m-1} \cos^{2m-1}\theta\, r^{2n-1} \sin^{2n-1}\theta\ e^{-r^2}\, r\, dr\, d\theta$$
$$= 4\int_0^{\pi/2} \int_0^\infty r^{2m+2n-2}\, e^{-r^2} \cos^{2m-1}\theta\, \sin^{2n-1}\theta\, r\, dr\, d\theta$$

Then, writing $w = r^2 \quad \therefore\ dw = 2r\, dr$

$$\Gamma(m)\Gamma(n) = 2\int_0^\infty w^{m+n-1}\, e^{-w}\, dw \int_0^{\pi/2} \sin^{2n-1}\theta\, \cos^{2m-1}\theta\, d\theta$$
$$= \Gamma(m+n) \times \mathrm{B}(m, n)$$

$$\therefore\ \mathrm{B}(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{13}$$

So $\mathrm{B}(\tfrac{3}{2}, \tfrac{1}{2}) = \ldots\ldots\ldots\ldots$

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$$\boxed{\mathrm{B}(\tfrac{3}{2}, \tfrac{1}{2}) = \frac{\pi}{2}}$$

Because

$$\mathrm{B}(\tfrac{3}{2}, \tfrac{1}{2}) = \frac{\Gamma(\tfrac{3}{2})\Gamma(\tfrac{1}{2})}{\Gamma(2)} = \frac{\sqrt{\pi}/2 \times \sqrt{\pi}}{1} = \frac{\pi}{2}$$

Now for some examples.

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Application of gamma and beta functions

The use of gamma and beta functions in the evaluation of definite integrals depends largely on the ability to change the variables to express the integral in the basic form of the beta function $\int_0^1 x^{m-1}(1-x)^{n-1}\mathrm{d}x$

or its trigonometrical form $2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\mathrm{d}\theta$.

Example 1

Evaluate $I = \int_0^1 x^5(1-x)^4\,\mathrm{d}x$.

Compare this with $\mathrm{B}(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\mathrm{d}x$

Then $m-1=5 \quad \therefore\ m=6$ and $n-1=4 \quad \therefore\ n=5$

$$\therefore\ I = \mathrm{B}(6, 5) = \ldots\ldots\ldots\ldots$$

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$$\boxed{I = \mathrm{B}(6, 5) = \frac{5!\,4!}{10!} = \frac{1}{1260}}$$

Example 2

Evaluate $I = \int_0^1 x^4\sqrt{1-x^2}\mathrm{d}x$.

Comparing this with $\mathrm{B}(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\mathrm{d}x$

we see that we have x^2 in the root, instead of a single x.

Therefore, put $x^2 = y \quad \therefore\ x = y^{\frac{1}{2}} \quad \mathrm{d}x = \frac{1}{2}y^{-\frac{1}{2}}\mathrm{d}y$

The limits remain unchanged. $\therefore\ I = \ldots\ldots\ldots\ldots$

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$$\boxed{I = \tfrac{1}{2}\mathrm{B}\left(\tfrac{5}{2}, \tfrac{3}{2}\right)}$$

Because

$$I = \int_0^1 y^2(1-y)^{\frac{1}{2}}\,\frac{1}{2}y^{-\frac{1}{2}}\mathrm{d}y = \frac{1}{2}\int_0^1 y^{\frac{3}{2}}(1-y)^{\frac{1}{2}}\mathrm{d}y$$

$m-1=\frac{3}{2} \quad \therefore\ m=\frac{5}{2}$ and $n-1=\frac{1}{2} \quad \therefore\ n=\frac{3}{2}$

$\therefore\ I = \frac{1}{2}\mathrm{B}\left(\frac{5}{2}, \frac{3}{2}\right)$

Expressing this in gamma functions

$$I = \ldots\ldots\ldots\ldots$$

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$$\boxed{I = \frac{1}{2}\,\frac{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{\Gamma(4)}}$$

From our previous work on gamma functions

$$\Gamma(\tfrac{3}{2}) = \frac{\sqrt{\pi}}{2};\ \ \Gamma(\tfrac{5}{2}) = \frac{3\sqrt{\pi}}{4};\ \ \Gamma(4) = 3!$$

$\therefore\ I = \ldots\ldots\ldots\ldots$

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$$\boxed{I = \frac{\pi}{32}}$$

Because

$$I = \frac{1}{2}\cdot\frac{(3\sqrt{\pi}/4)(\sqrt{\pi}/2)}{3!} = \frac{\pi}{32}.$$

Now you can work through this one in much the same way. There are no tricks.

Example 3

Evaluate $I = \displaystyle\int_0^3 \frac{x^3\,dx}{\sqrt{3-x}}$.

You need to compare this with $B(m, n) = \displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ so bring everything up on to the top line and then make the necessary change in the variables. Finish it off and then compare the results with the next frame.

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$$\boxed{I = \frac{864\sqrt{3}}{35} = 42{\cdot}76}$$

Here is the working; see whether you agree.

$$I = \int_0^3 \frac{x^3\,dx}{\sqrt{3-x}} = \int_0^3 x^3(3-x)^{-\frac{1}{2}}\,dx = 3^{-\frac{1}{2}}\int_0^3 x^3\left(1-\frac{x}{3}\right)^{-\frac{1}{2}}dx$$

Put $\dfrac{x}{3} = y$, i.e. $x = 3y$ $\quad\therefore\ dx = 3\,dy$

Limits: $x = 0,\ y = 0;\quad x = 3,\ y = 1$

$$\therefore\ I = 27\sqrt{3}\int_0^1 y^3(1-y)^{-\frac{1}{2}}\,dy \qquad \begin{array}{ll} m-1 = 3 & \therefore\ m = 4 \\ n-1 = -\frac{1}{2} & \therefore\ n = \frac{1}{2} \end{array}$$

$$\therefore\ I = 27\sqrt{3}\,B(4, \tfrac{1}{2}) = 27\sqrt{3}\,\frac{\Gamma(4)\Gamma(\frac{1}{2})}{\Gamma(9/2)}$$

Now $\Gamma(\frac{1}{2}) = \sqrt{\pi};\ \Gamma(9/2) = \dfrac{105\sqrt{\pi}}{16};\ \Gamma(4) = 3!$

$$\therefore\ I = 27\sqrt{3}\times 6\times\sqrt{\pi}\times\frac{16}{105\sqrt{\pi}} = \frac{864\sqrt{3}}{35} = 42{\cdot}76$$

▶

Example 4

Evaluate $I = \int_0^{\pi/2} \sin^5\theta \cos^4\theta \, d\theta$.

$$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta$$

$$\therefore\ 2m - 1 = 5 \quad \therefore\ m = 3; \quad 2n - 1 = 4 \quad \therefore\ n = 5/2$$

$$\therefore\ I = \tfrac{1}{2} B(3, 5/2) = \ldots\ldots\ldots\ldots$$

Finish it off

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$$\boxed{I = \frac{8}{315}}$$

$$I = \frac{1}{2} B(3, 5/2) = \frac{1}{2}\cdot\frac{\Gamma(3)\Gamma(5/2)}{\Gamma(11/2)}$$

$$= \frac{1}{2}\cdot\frac{2!(3\sqrt{\pi})/4}{(945\sqrt{\pi})/32} = \frac{3\sqrt{\pi}}{4}\cdot\frac{32}{945\sqrt{\pi}} = \frac{8}{315}$$

Finally, one more.

Example 5

Evaluate $I = \int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta$.

Somehow, we need to turn this into the form

$$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta$$

So off you go; express the result in gamma functions

$$I = \ldots\ldots\ldots\ldots$$

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$$\boxed{I = \frac{1}{2}\cdot\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{\Gamma(1)}}$$

Because

$$I = \int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta = \int_0^{\pi/2} \sin^{\frac{1}{2}}\theta \cos^{-\frac{1}{2}}\theta \, d\theta$$

$$\therefore\ 2m - 1 = \frac{1}{2} \quad \therefore\ m = \frac{3}{4}; \quad 2n - 1 = -\frac{1}{2} \quad \therefore\ n = \frac{1}{4}$$

$$\therefore\ I = \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{1}{2}\cdot\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{\Gamma(1)}$$

and, unless we have appropriate tables to evaluate $\Gamma(\frac{3}{4})$ and $\Gamma(\frac{1}{4})$, we cannot proceed much further. However, we do have such a table in Frame 24 so refer to it to evaluate the integral of our example.

$$I = \ldots\ldots\ldots\ldots$$

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$$\boxed{I = 2{\cdot}2214}$$

Because

$$\Gamma(0{\cdot}25) = 3{\cdot}6256 \quad \text{and} \quad \Gamma(0{\cdot}75) = 1{\cdot}2254$$

$$\therefore \; I = \frac{1}{2} \cdot \frac{(1{\cdot}2254)(3{\cdot}6256)}{1{\cdot}0000} = 2{\cdot}2214$$

Duplication formula for gamma functions

We already know that, when n is a positive integer

$$\Gamma(n) = (n-1)!$$

A useful formula enables us to calculate the gamma functions for values of n halfway between the integers. This is the *duplication formula* which can be stated as

$$\Gamma(n + \tfrac{1}{2}) = \frac{\Gamma(2n)\sqrt{\pi}}{2^{2n-1}\Gamma(n)} \tag{14}$$

Thus, to find $\Gamma(3{\cdot}5)$ $\quad \Gamma(n) = \Gamma(3) = 2!$

$$\Gamma(2n) = \Gamma(6) = 5!$$

$$\therefore \; \Gamma(3{\cdot}5) = \Gamma(3 + \tfrac{1}{2}) = \frac{5!\sqrt{\pi}}{2^5\,2!} = 3{\cdot}3234$$

The formula is quoted here without proof, but it is useful to have on occasions.

So $\Gamma(6{\cdot}5) = \ldots\ldots\ldots\ldots$

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$$\boxed{\Gamma(6{\cdot}5) = 287{\cdot}9}$$

$$\Gamma(6{\cdot}5) = \Gamma(6 + \tfrac{1}{2}) = \frac{\Gamma(12)\sqrt{\pi}}{2^{11}\Gamma(6)}$$

$$\Gamma(6) = 5!; \;\; \Gamma(12) = 11!; \;\; 2^{11} = 2048$$

$$\therefore \; \Gamma(6{\cdot}5) = \frac{11!\sqrt{\pi}}{2048 \times 5!} = 287{\cdot}9$$

Now let us consider another function represented by an integral.

On then to the next frame

The error function

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The error function erf (x) is defined as

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,\mathrm{d}t$$

and occurs in statistics and various studies in physics and engineering. This integral, for arbitrary x, can only be evaluated numerically and values of erf (x) for various values of x are obtained from tables.

Where the limits of $\int_a^b e^{-t^2}\,\mathrm{d}t$ are zero or $\pm\infty$, however, an exact result is possible. We have already considered the integral $I=\int_0^\infty e^{-t^2}\,\mathrm{d}t$ in Frame 6 when dealing with gamma functions and we established then that

$$\int_0^\infty e^{-t^2}\,\mathrm{d}t=\ldots\ldots\ldots\ldots$$

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$$\boxed{\int_0^\infty e^{-t^2}\,\mathrm{d}t=\frac{1}{2}\Gamma\left(\tfrac{1}{2}\right)=\frac{\sqrt{\pi}}{2}}$$

Consequently

$$\underset{x\to\infty}{Lim}(\operatorname{erf}(x))=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\,\mathrm{d}t=1$$

By representing the exponential function in the integral by its Maclaurin series we see that

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\ldots\ldots\ldots\ldots$$

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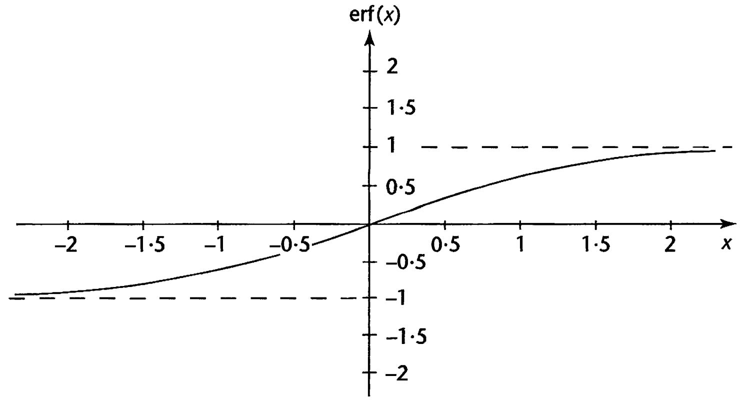
$$\boxed{\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)}}$$

Because

$$\begin{aligned}\operatorname{erf}(x)&=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,\mathrm{d}t\\&=\frac{2}{\sqrt{\pi}}\int_0^x\left(\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{n!}\right)\mathrm{d}t\\&=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\left(\int_0^x\frac{(-1)^n t^{2n}}{n!}\,\mathrm{d}t\right)\\&=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)}\end{aligned}$$

Consequently erf $(-x)=-$erf (x) and so erf (x) is an odd function. ▶

The graph of erf(*x*)



The complementary error function erfc(*x*)

The complementary error function is defined as

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\,\mathrm{d}t$$

which is related to the error function by the relation

$$\operatorname{erfc}(x) = \ldots\ldots\ldots\ldots$$

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$$\boxed{\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)}$$

Because

$$\begin{aligned}\operatorname{erfc}(x) &= \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\,\mathrm{d}t \\ &= \frac{2}{\sqrt{\pi}} \left(\int_0^{\infty} e^{-t^2}\,\mathrm{d}t - \int_0^{x} e^{-t^2}\,\mathrm{d}t \right) \\ &= 1 - \operatorname{erf}(x)\end{aligned}$$

Example 1

In terms of the complementary error function, for $0 < a < b$

$$\int_a^b e^{-t^2}\,\mathrm{d}t = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{\sqrt{\pi}}{2}[\operatorname{erfc}(a) - \operatorname{erfc}(b)]}$$

Because

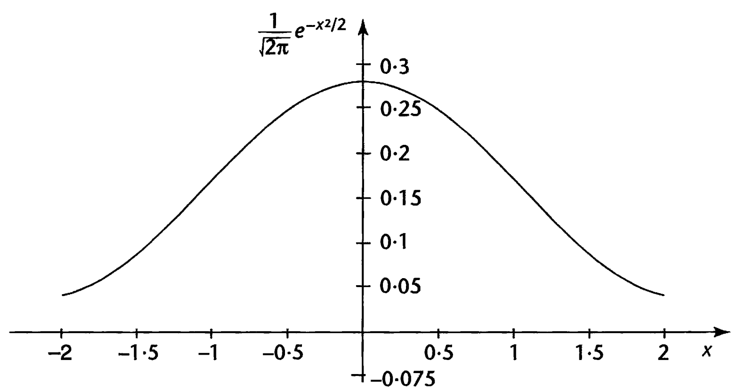
$$\begin{aligned}\int_a^b e^{-t^2}\,dt &= \int_0^b e^{-t^2}\,dt - \int_0^a e^{-t^2}\,dt\\ &= \frac{\sqrt{\pi}}{2}\operatorname{erf}(b) - \frac{\sqrt{\pi}}{2}\operatorname{erf}(a)\\ &= \frac{\sqrt{\pi}}{2}[1-\operatorname{erfc}(b)] - \frac{\sqrt{\pi}}{2}[1-\operatorname{erfc}(a)]\\ &= \frac{\sqrt{\pi}}{2}[\operatorname{erfc}(a) - \operatorname{erfc}(b)]\end{aligned}$$

Example 2

In statistics the integral

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt$$

is the area beneath the Gaussian or normal probability distribution $\dfrac{1}{\sqrt{2\pi}}e^{-t^2/2}$ for the values $-\infty < t \le x$.



The area beneath the complete Gaussian curve is then

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}\,dt = \ldots\ldots\ldots\ldots$$

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$$\boxed{1}$$

Because

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}\,\mathrm{d}t &= \frac{1}{\sqrt{2\pi}}\left(2\int_0^{\infty} e^{-t^2/2}\,\mathrm{d}t\right) && \text{because the integrand is even}\\ &= \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-t^2/2}\,\mathrm{d}t\\ &= \sqrt{\frac{2}{\pi}}\times\sqrt{2}\int_0^{\infty} e^{-u^2}\,\mathrm{d}u && \text{where } u = t/\sqrt{2}\\ &= 1 && \text{from Frame 47}\end{aligned}$$

For positive x, $\Phi(x)$ is related to the error function

$$\Phi(x) = \dots\dots\dots\dots$$

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$$\boxed{\Phi(x) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)}$$

Because

$$\begin{aligned}\Phi(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,\mathrm{d}t\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-t^2/2}\,\mathrm{d}t + \frac{1}{\sqrt{2\pi}}\int_0^{x} e^{-t^2/2}\,\mathrm{d}t\\ &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\times\sqrt{2}\int_0^{x/\sqrt{2}} e^{-u^2}\,\mathrm{d}u \quad \text{where } u = t/\sqrt{2}\\ &= \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\end{aligned}$$

Now let us consider a new set of integral funtions.

Elliptic functions

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The use of *elliptic functions* provides a means of evaluating a further range of definite integrals, provided that the integrals can be converted by various appropriate substitutions into certain standard forms.

If an integrand is a rational function of x and of $\sqrt{P(x)}$ where $P(x)$ is a polynomial in x of degree 3 or 4, then the integral is said to be *elliptic*.

For example, $\displaystyle\int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-2x^2)(4-3x^2)}}$ is an elliptic integral. The name is derived from such an integral occurring in the determination of the arc length of part of an ellipse.

▶

Standard forms of elliptic functions

(a) *Of the first kind*

$$F(k, \phi) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \tag{1}$$

where $0 \leq \phi \leq \pi/2$ and $0 < k < 1$.

(b) *Of the second kind*

$$E(k, \phi) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta \tag{2}$$

where $0 \leq \phi \leq \frac{\pi}{2}$ and $0 < k < 1$.

Make a careful note of these two standard forms: then we can apply them to some examples.

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Example 1

Evaluate $\int_0^{\pi/2} \sqrt{4 - \sin^2 \theta}\, d\theta$ in terms of an elliptic function.

Taking out a factor 4 to reduce the first term to 1

$$I = 2\int_0^{\pi/2} \sqrt{1 - \frac{1}{4}\sin^2 \theta}\, d\theta$$

The integral now agrees with the standard form, where $k^2 = \frac{1}{4}$, i.e. $k = \frac{1}{2}$ and $\phi = \pi/2$.

$$\therefore\ I = \ldots\ldots\ldots\ldots$$

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$$\boxed{I = 2E(\tfrac{1}{2}, \pi/2)}$$

Complete elliptic functions

In each of the cases (1) and (2) listed above, if $\phi = \pi/2$, the integral is said to be *complete* and then

$F(k, \pi/2)$ is denoted by $K(k)$

and $E(k, \pi/2)$ is denoted by $E(k)$.

The method, then, rests on making suitable substitutions in a given integral to transform the integrand into one of the standard forms stated above. For various values of k and ϕ, values of the functions $F(k, \phi)$, $E(k, \phi)$, $K(k)$ and $E(k)$ are obtainable from published tables. These tables, which are quite extensive, are not reproduced here and so many required values will be given in the text.

Incidentally, the result of Example 1 above, i.e. $I = 2E(\frac{1}{2}, \pi/2)$ could also be written as

$$I = \ldots\ldots\ldots\ldots$$

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$$\boxed{I = 2E(\tfrac{1}{2})}$$

because, in this case, $\phi = \pi/2$.

From tables, we find that $E(\frac{1}{2}) = 1{\cdot}4675 \quad \therefore\ I = 2{\cdot}935$

Example 2

Evaluate $I = \displaystyle\int_0^{\pi/6} \frac{d\theta}{\sqrt{1 - 4\sin^2\theta}}$.

At first sight, this seems to be in standard form, but notice that the value of k^2 is 4, i.e. $k = 2$ – and this does not comply with the requirement that $0 < k < 1$. We therefore proceed as follows.

$$I = \int_0^{\pi/6} \frac{d\theta}{\sqrt{1 - 4\sin^2\theta}}$$

Put $\quad 4\sin^2\theta = \sin^2\psi$

i.e. $\quad 2\sin\theta = \sin\psi$

$$\therefore\ 2\cos\theta\, d\theta = \cos\psi\, d\psi \quad \therefore\ d\theta = \frac{\cos\psi\, d\psi}{2\cos\theta}$$

Also, for the new limits, when $\theta = 0$, $\quad \psi = \dots\dots\dots\dots$

and $\quad$ when $\theta = \pi/6$, $\quad \psi = \dots\dots\dots\dots$

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$$\boxed{\theta = 0,\ \ \psi = 0; \quad \theta = \pi/6,\ \ \psi = \pi/2}$$

$$\therefore\ I = \int_0^{\pi/2} \frac{1}{\sqrt{1 - \sin^2\psi}} \cdot \frac{\cos\psi\, d\psi}{2\ \cos\theta}$$

We now transform the $\cos\theta$

$$\sin\theta = \tfrac{1}{2}\sin\psi \quad \therefore\ 1 - \cos^2\theta = \tfrac{1}{4}\sin^2\psi \quad \therefore\ \cos\theta = \sqrt{1 - \tfrac{1}{4}\sin^2\psi}$$

$$\therefore\ I = \frac{1}{2}\int_0^{\pi/2} \frac{1}{\cos\psi} \cdot \frac{\cos\psi\, d\psi}{\sqrt{1 - \frac{1}{4}\sin^2\psi}}$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - \frac{1}{4}\sin^2\psi}} \quad \text{which is now in standard form}$$

$$\therefore\ I = \dots\dots\dots\dots$$

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$$\boxed{I = \tfrac{1}{2}F(\tfrac{1}{2}, \pi/2) = \tfrac{1}{2}K(\tfrac{1}{2})}$$

From the appropriate tables, $K(\frac{1}{2}) = 1{\cdot}6858 \quad \therefore \ I = 0{\cdot}8429$
Now for another

Example 3

Evaluate $I = \displaystyle\int_0^{\pi/3} \frac{d\theta}{\sqrt{3 - 4\sin^2\theta}}$.
The first step is to ....

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take out a factor 3 to reduce the first term to 1

$$\therefore \ I = \frac{1}{\sqrt{3}}\int_0^{\pi/3} \frac{d\theta}{\sqrt{1 - \frac{4}{3}\sin^2\theta}}$$

Next, we see that $k^2 > 1$. Therefore, we put ....

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$$\boxed{\tfrac{4}{3}\sin^2\theta = \sin^2\psi}$$

$$\frac{2}{\sqrt{3}}\sin\theta = \sin\psi \quad \therefore \quad \frac{2}{\sqrt{3}}\cos\theta \, d\theta = \cos\psi \, d\psi \quad \therefore \ d\theta = \frac{\sqrt{3}\cos\psi \, d\psi}{2\cos\theta}$$

Then, so far, we have $I =$ ....

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$$\boxed{I = \frac{1}{\sqrt{3}}\int_{\theta=0}^{\theta=\pi/3} \frac{1}{\sqrt{1 - \sin^2\psi}} \cdot \frac{\sqrt{3}\cos\psi \, d\psi}{2\cos\theta}}$$

$$\frac{2}{\sqrt{3}}\sin\theta = \sin\psi$$

Limits: when $\theta = 0$, $\psi = 0$

$$\theta = \frac{\pi}{3}, \ \frac{2}{\sqrt{3}}\sin\theta = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = 1 \quad \therefore \ \psi = \pi/2$$

Also $\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \frac{3}{4}\sin^2\psi}$

$$\therefore \ I = \text{...........}$$

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$$I = \frac{1}{2}\int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - \frac{3}{4}\sin^2\psi}}$$

which is now in standard form with $k = \dfrac{\sqrt{3}}{2}$ and $\phi = \pi/2$

$$\therefore\ I = \frac{1}{2}F\left(\frac{\sqrt{3}}{2},\ \pi/2\right) = \frac{1}{2}K\left(\frac{\sqrt{3}}{2}\right)$$

From tables $K\left(\dfrac{\sqrt{3}}{2}\right) = 2{\cdot}1565 \qquad \therefore\ I = 1{\cdot}078$

Now, what about this one?

Example 4

Evaluate $I = \displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 + 4\sin^2\theta}}$.

The trouble here is the *plus* sign in the denominator. Were it a minus sign as in Example 2, the integral could be converted into standard form and would present no difficulty.

In this case, the key is to put $\theta = \pi/2 - \psi$, i.e. $\sin\theta = \cos\psi$.

Expressing the integral in terms of ψ, we have

$I = \ldots\ldots\ldots\ldots$

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$$I = \int_{\pi/2}^{0} \frac{-d\psi}{\sqrt{5 - 4\sin^2\psi}}$$

Because

$\theta = \pi/2 - \psi \qquad \therefore\ d\theta = -d\psi$

$1 + 4\sin^2\theta = 1 + 4(1 - \cos^2\theta) = 5 - 4\cos^2\theta = 5 - 4\sin^2\psi$

Limits: when $\theta = 0$, $\psi = \pi/2$; when $\theta = \pi/2$, $\psi = 0$ and the expression above immediately follows.

Move on

64

So we have $I = \displaystyle\int_{\pi/2}^{0} \frac{-d\psi}{\sqrt{5 - 4\sin^2\psi}}$

The minus sign in the numerator can be absorbed by $\ldots\ldots\ldots\ldots$

65

$$\boxed{\text{changing the order of the limits}}$$

$$\therefore\ I = \int_0^{\pi/2} \frac{d\psi}{\sqrt{5 - 4\sin^2\psi}}$$

Finally, taking out a factor 5 from the denominator, the integral becomes

$$I = \frac{1}{\sqrt{5}} \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - \frac{4}{5}\sin^2\psi}}$$

and this can then be written ....

66

$$\boxed{I = \frac{1}{\sqrt{5}} F\left(\frac{2}{\sqrt{5}}, \frac{\pi}{2}\right) = \frac{1}{\sqrt{5}} K\left(\frac{2}{\sqrt{5}}\right)}$$

From tables $K\left(\frac{2}{\sqrt{5}}\right) = K(0{\cdot}8944) = 2{\cdot}2435 \quad \therefore\ I = 1{\cdot}003$

Alternative forms of elliptic functions

(a) *Of the first kind*

$$F(k, x) = \int_0^x \frac{du}{\sqrt{(1 - u^2)(1 - k^2u^2)}} \tag{3}$$

where $0 \leq x \leq 1$ and $0 < k < 1$.

(b) *Of the second kind*

$$E(k, x) = \int_0^x \sqrt{\frac{1 - k^2u^2}{1 - u^2}}\, du \tag{4}$$

where $0 \leq x \leq 1$ and $0 < k < 1$.

Note these two new forms and then we can deal with a few examples. As before, it is a case of transforming the given integrand into the required form by suitable substitutions.

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Example 1

Evaluate $I = \displaystyle\int_0^{1/\sqrt{2}} \sqrt{\frac{4 - 3u^2}{1 - u^2}}\, du.$

Here we remove a factor 4 from the numerator to reduce the first term to 1.

$$I = 2\int_0^{1/\sqrt{2}} \sqrt{\frac{1 - \frac{3}{4}u^2}{1 - u^2}}\, du$$

This is now in standard form with $k =$ .... and $x =$ ....

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$$\boxed{k = \frac{\sqrt{3}}{2}; \quad x = \frac{1}{\sqrt{2}}}$$

$$\therefore\ I = 2E\left(\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}\right) = 2(0{\cdot}7282) \text{ from tables}$$

$$\therefore\ I = 1{\cdot}4564$$

Example 2

Evaluate $I = \int_0^{1/2} \frac{\mathrm{d}u}{\sqrt{5 - 6u^2 + u^4}}$.

Factorising the denominator gives $I = \ldots\ldots\ldots\ldots$

69

$$\boxed{I = \int_0^{1/2} \frac{\mathrm{d}u}{\sqrt{(1-u^2)(5-u^2)}}}$$

Taking out a factor 5

$$I = \frac{1}{\sqrt{5}} \int_0^{1/2} \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-\frac{1}{5}u^2)}}$$

which is in standard form with $k = 1/\sqrt{5}$ and $x = 1/2$

$$\therefore\ I = \ldots\ldots\ldots\ldots$$

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$$\boxed{I = \frac{1}{\sqrt{5}} F\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)}$$

In some tables, k is quoted as $\sin\theta$, i.e. $\sin\theta = \frac{1}{\sqrt{5}}$ $\quad\therefore\ \theta = 26°\ 34'$

and x is quoted as $\sin\phi$, i.e. $\sin\phi = \frac{1}{2}$ $\quad\therefore\ \phi = 30°$.

Then $F(1/\sqrt{5}, 1/2) = 0{\cdot}528$

$\therefore\ I = 0{\cdot}236$

Now move on for Example 3

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Example 3

Evaluate $I = \int_0^{\sqrt{3}/4} \sqrt{\frac{2-x^2}{1-4x^2}}\,dx$.

We have to convert this into the form $\int \sqrt{\frac{1-k^2u^2}{1-u^2}}\,du$, so first concentrate on the denominator. Any suggestions?

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$$\boxed{\text{Put } 4x^2 = u^2 \quad \text{i.e. } 2x = u}$$

$4x^2 = u^2 \quad \therefore\ 2x = u \quad \therefore\ 2\,dx = du$

Limits: when $x = 0$, $u = 0$ and when $x = \sqrt{3}/4$, $u = \sqrt{3}/2$

Also $\quad 2 - x^2 = 2 - u^2/4$

The integral now becomes ....

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$$\boxed{I = \int_0^{\sqrt{3}/2} \sqrt{\frac{2-u^2/4}{1-u^2}} \cdot \frac{du}{2}}$$

Finally, taking out the factor 2 in the numerator

$I = \ldots\ldots\ldots\ldots$

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$$\boxed{I = \frac{1}{\sqrt{2}} \int_0^{\sqrt{3}/2} \frac{\sqrt{1-u^2/8}}{1-u^2}\,du}$$

i.e. $k^2 = \frac{1}{8} \quad \therefore\ k = \frac{\sqrt{2}}{4}$ and $x = \frac{\sqrt{3}}{2}$

So $I = \ldots\ldots\ldots\ldots$

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$$\boxed{I = \frac{1}{\sqrt{2}} E\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{3}}{2}\right)}$$

Then $\sin\theta = \frac{\sqrt{2}}{4} \quad \therefore\ \theta = 20^\circ\,42'$ and $\sin\phi = \frac{\sqrt{3}}{2} \quad \therefore\ \phi = 60^\circ$

From tables, $E\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{3}}{2}\right) = 1{\cdot}029 \quad \therefore\ I = 0{\cdot}728$

So it is all just a question of manipulation to transform the given integral into the required standard forms, and then of reference to the appropriate tables.

▶

The **Revision summary** follows, to be read in conjunction with the **Can You?** checklist, checking with the relevant parts of the Programme any points of which you are unsure. You will then find the **Test exercise** straightforward. Finally the **Further problems** provide additional practice.

Revision summary 16

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1 *Gamma functions*

(a) $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \qquad x > 0$

$\Gamma(x+1) = x\Gamma(x)$

(b) If $x = n$, a positive integer

$\Gamma(n+1) = n!$

$\Gamma(1) = 1$

$\Gamma(0) = \infty \qquad \Gamma(-n) = \pm\infty$

(c) $\int_0^\infty e^{-x^2}\,dx = \dfrac{\sqrt{\pi}}{2}$

(d) $\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \qquad \Gamma(\tfrac{3}{2}) = \dfrac{\sqrt{\pi}}{2}$

$\Gamma(\tfrac{5}{2}) = \dfrac{3\sqrt{\pi}}{4} \qquad \Gamma(\tfrac{7}{2}) = \dfrac{15\sqrt{\pi}}{8}$

$\Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi} \qquad \Gamma(-\tfrac{3}{2}) = \dfrac{4\sqrt{\pi}}{3}$

(e) Duplication formula $\Gamma\left(n+\tfrac{1}{2}\right) = \dfrac{\Gamma(2n)\sqrt{\pi}}{2^{2n-1}\cdot\Gamma(n)}$

2 *Beta functions*

(a) $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad m > 0;\ n > 0$

$B(m, n) = B(n, m)$

$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$

(b) $B(m, n) = \dfrac{(m-1)(n-1)}{(m+n-1)(m+n-2)}B(m-1, n-1)$

$B(k, 1) = B(1, k) = \dfrac{1}{k}$

$B(1, 1) = 1; \qquad B(\tfrac{1}{2}, \tfrac{1}{2}) = \pi$

$B(m, n) = \dfrac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$

(c) m and n positive integers

$B(m, n) = \dfrac{(m-1)!(n-1)!}{(m+n-1)!}$

▶

3 *Error function*

(a) $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2}\,dt$

(b) $\displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}; \qquad \int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

Complementary error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt = 1 - \operatorname{erf}(x)$$

4 *Elliptic functions*

(a) *Standard forms*

(1) of the first kind: $F(k, \phi) = \displaystyle\int_0^\phi \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}$

(2) of the second kind: $E(k, \phi) = \displaystyle\int_0^\phi \sqrt{1 - k^2\sin^2\theta}\,d\theta$

In each case, $0 \le \phi \le \pi/2$; $0 < k < 1$.

(b) *Complete elliptic integrals* $\phi = \dfrac{\pi}{2}$

$$F\left(k, \frac{\pi}{2}\right) = K(k)$$
$$E\left(k, \frac{\pi}{2}\right) = E(k)$$

(c) *Alternative forms of elliptic functions*

(1) of the first kind: $F(k, x) = \displaystyle\int_0^x \frac{du}{\sqrt{(1 - u^2)(1 - k^2u^2)}}$

(2) of the second kind: $E(k, x) = \displaystyle\int_0^x \sqrt{\frac{1 - k^2u^2}{1 - u^2}}\,du$

In each case $0 \le x \le 1$; $0 < k < 1$.

☑ Can You?

Checklist 16

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Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Derive the recurrence relation for the gamma function and evaluate the gamma function for certain rational arguments? 1 to 16

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Evaluate integrals that require the use of the gamma function in their solution? 17 to 24

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Identify the beta function and evaluate integrals that require the use of the beta function in their solution? 25 to 32

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Derive the relationship between the gamma function and the beta function? 33 to 44

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use the duplication formula to evaluate the gamma function for half integer arguments? 44 and 45

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Recognise the error function and its relation to the Gaussian probability distribution? 46 to 52

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Recognise elliptic functions of the first and second kind? 53

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Evaluate integrals that require the use of elliptic functions in their solution? 54 to 66

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use alternative forms of the elliptic functions? 66 to 75

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 16

78

1 Evaluate (a) $\dfrac{\Gamma(6)}{3\Gamma(4)}$ (b) $\dfrac{\Gamma(1{\cdot}5)}{\Gamma(2{\cdot}5)}$ (c) $\dfrac{\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{2})}$

(d) $\displaystyle\int_0^\infty x^5\,e^{-x}\,dx$ (e) $\displaystyle\int_0^\infty x^6\,e^{-4x^2}\,dx.$

2 Determine (a) $\displaystyle\int_0^1 x^5(2-x)^4 dx$

(b) $\displaystyle\int_0^{\pi/2} \sin^7\theta\cos^3\theta\,d\theta$

(c) $\displaystyle\int_0^{\pi/8} \sin^2 4\theta\cos^5 4\theta\,d\theta.$

3 Show that

(a) $\displaystyle\int_{-a}^{a} e^{-t^2}\,dt = \sqrt{\pi}\,\mathrm{erf}\,(a)$

(b) $\displaystyle\int_0^\infty e^{-k^2t^2}\,dt = \frac{\sqrt{\pi}}{2k},\quad k>0.$

4 Evaluate

(a) erfc (∞)

(b) erfc (0).

5 Express the following in elliptic functions.

(a) $\displaystyle\int_0^{\pi/4} \frac{d\theta}{\sqrt{1-2\sin^2\theta}}$

(b) $\displaystyle\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{4-5u^2+u^4}}.$

Further problems 16

79

1 Evaluate (a) $\dfrac{\Gamma(5)}{2\Gamma(3)}$; (b) $\dfrac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{1}{2})}$; (c) $\dfrac{\Gamma(2{\cdot}5)}{\Gamma(3{\cdot}5)}$;

(d) $\displaystyle\int_0^\infty x^4e^{-x}\,dx$; (e) $\displaystyle\int_0^\infty x^8e^{-2x}\,dx.$

2 Determine (a) $\displaystyle\int_0^\infty x^3e^{-x}\,dx$; (b) $\displaystyle\int_0^\infty x^4e^{-3x}\,dx$;

(c) $\displaystyle\int_0^\infty x^2e^{-2x^2}dx$; (d) $\displaystyle\int_0^\infty \sqrt{x}\cdot e^{-\sqrt{x}}\,dx.$

3 If m and n are positive constants, show that $\int_0^{\infty} x^m e^{-ax^n}\,dx$ can be expressed in the form $\dfrac{1}{n \cdot a^{(m+1)/n}}\Gamma\left(\dfrac{m+1}{n}\right)$.

4 Evaluate the following.

(a) $\int_0^{1/2} x^4(1-2x)^3\,dx$

(b) $\int_0^{1/\sqrt{2}} x^2\sqrt{1-2x^2}\,dx$

(c) $\int_0^{\pi/2} \sin^5\theta\cos^4\theta\,d\theta$

(d) $\int_0^{\pi/2} \sin\theta\sqrt{\cos^5\theta}\,d\theta$

(e) $\int_0^{\pi/4} \sin^3 2\theta\cos^6 2\theta\,d\theta$

(f) $\int_0^{1/3} x^2\sqrt{1-9x^2}\,dx$.

5 Show that $\dfrac{d}{dx}\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}}e^{-x^2}$.

6 Show that the Laplace transform of the error function is given as

$$F(s) = \int_0^{\infty} \operatorname{erf}(t)\,e^{-st}\,dt = \frac{e^{-s^2/4}}{s}\operatorname{erfc}\left(\frac{s}{2}\right) \text{ for } s > 0.$$

7 The Fresnel integrals are defined as

$$C(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right)dt \text{ and } S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right)dt$$

Show that

$$\frac{1}{\sqrt{2j}}\operatorname{erf}\left(x\sqrt{\frac{j\pi}{2}}\right) = C(x) - jS(x)$$

8 Express the following in elliptic functions.

(a) $\displaystyle\int_0^{\pi/2} \sqrt{1+4\sin^2\theta}\,d\theta$

(b) $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta}}$

(c) $\displaystyle\int_0^{1} \sqrt{\frac{4-x^2}{1-x^2}}\,dx$

(d) $\displaystyle\int_0^{2} \frac{dx}{\sqrt{(9-x^2)(16-x^2)}}$

(e) $\displaystyle\int_0^{2} \frac{dx}{\sqrt{(4-x^2)(5-x^2)}}$

(f) $\displaystyle\int_0^{\pi/6} \frac{d\theta}{\sqrt{\sin^2\theta+2\cos^2\theta}}$

(g) $\displaystyle\int_{\pi/4}^{\pi/3} \frac{d\theta}{\sqrt{\sin^2\theta+2\cos^2\theta}}.$

9 Using the substitution $x=\tan\theta$ prove that the integral

$$\int_0^1 \frac{dx}{\sqrt{(1+x^2)(1+4x^2)}}$$

can be expressed in the form

$$\frac{1}{2}\int_0^{\pi/4} \frac{d\theta}{\sqrt{1-\frac{3}{4}\cos^2\theta}}$$

Hence, using $\theta=\dfrac{\pi}{2}-\phi$, evaluate the integral in terms of elliptic functions.

10 Evaluate the following.

(a) $\displaystyle\int_0^{0\cdot5} \frac{dx}{\sqrt{3-4x^2+x^4}}$

(b) $\displaystyle\int_{0\cdot5}^{1\cdot0} \frac{dx}{\sqrt{3-4x^2+x^4}}$

(c) $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{25+9\sin^2\theta}}$

(d) $\displaystyle\int_0^{\pi/3} \frac{d\theta}{\sqrt{4+3\sin^2\theta}}.$